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Model of a quantum particle in spacetime

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Abstract

In their paper Doplicher, Fredenhagen and Roberts (DFR) proposed a simple model of a particle in quantum spacetime. We give a new formulation of the model and propose some small changes and additions which improve the physical interpretation. In particular, we show that the internal degrees of freedom e and m of the particle represent external forces acting on the particle. To obtain this result we follow a constructive approach. The model is formulated as a covariance system. It has projective representations in which not only the spacetime coordinates but also the conjugated momenta are two-by-two noncommuting. These momenta are of the form $P_\mu - (b/c)A_\mu$, where b is the charge of the particle. The electric and magnetic fields obtained from the vector potential A_μ coincide with the variables e and m postulated by DFR. Similarly, the spacetime position operators are of the form $Q_\mu - (al^2/\hbar c)\Omega_\mu$, where a is a generalized charge and l a fundamental length, and with vector potentials Ω_μ which are in some sense dual w.r.t. the A_μ.

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1. Introduction

The DFR model, introduced in 1994 by Doplicher, Fredenhagen and Roberts [5, 7], describes a relativistic quantum particle with internal degrees of freedom e and m which behave under Lorentz transformations as electric and magnetic field vectors respectively. Many authors have tried to describe the electron in terms of such internal degrees of freedom—see e.g. the classification of [3]. The DFR model is one of the simplest models of quantum spacetime, and as such has received a lot of attention in the literature. For example, [9] adopt the basic assumption of the model that the time–position commutators $[Q_\mu, Q_\nu]$ commute with all observables.

In previous work [10, 11] we have reformulated the model as a covariance system. It is common to study a quantum system starting from a Lie group X of relevant symmetries. In a covariance system this group is supplemented with an algebra of observables which transform into each other under the actions of the symmetry group. We expect that all models of quantum mechanics and quantum field theory can be described as covariance systems. For example, standard quantum mechanics of a nonrelativistic particle can be described as a covariance system consisting of an algebra of functions of position together with the Euclidean group—see [11].

In this covariance approach it is important to remember that unitary representations of a symmetry group are allowed to be projective. In particular, for the model under study, nonvanishing time–position commutators $[Q_0, Q_\alpha] \neq 0$, $\alpha = 1, 2, 3$, and nonvanishing momentum commutators $[P_\mu, P_\nu] \neq 0$ are obtained by considering projective representations of the group of shifts in spacetime and in momentum space. Indeed, let p and p' be two shifts in momentum space, and let U denote the projective representation. Then a phase factor ξ is allowed in the composition

$$U(p)U(p') = \xi(p, p')U(p + p'). \tag{1}$$

Now, write ξ in the form

$$\xi(p, p') = \exp\left(\frac{\mathrm{i}}{2}\sum_{\mu,\nu=0}^{3} p_\mu \mathcal{Q}_{\mu,\nu} p'_\nu\right) \tag{2}$$

with $\mathcal{Q}$ an anti-symmetric matrix. The time–position operators Q_μ are the generators of shifts in momentum space

$$U(p) = \exp\left(-\mathrm{i}\hbar^{-1}\sum_{\mu=0}^{3} p_\mu g_{\mu,\mu} Q_\mu\right) \tag{3}$$

(the metric tensor g is diagonal with eigenvalues 1, -1, -1 and -1). Combination of (3) and (1) implies the following commutation relations:

$$[Q_\mu, Q_\nu] = -\mathrm{i}\hbar^2 g_{\mu,\mu}\mathcal{Q}_{\mu,\nu} g_{\nu,\nu}. \tag{4}$$

In the DFR model the rhs of the latter expression is an operator which commutes with all other observables. Hence it is clear that also the phase factor $\xi(p, p')$ in (1) should be allowed to be an operator. Unitary representations with operator-valued phase factors have been studied in [10]. From a physical point of view they are acceptable if they correspond to gauge freedoms of the model, in other words, if the wavefunctions ψ and $\xi(p, p')\psi$ describe the same state of the system. This is obviously the case if $\xi(p, p')$ commutes with all observables.

Small changes of and additions to the original DFR model are necessary to clarify the structure of the model. In the present paper we limit ourselves to the description of a single particle. In [5, 7] also fields are considered. The technicality of the latter makes it hard to analyse the field version with the same depth as is possible for the single-particle version. The drawback of the present approach is that we are forced to use the rather uncommon off-shell formalism of relativistic quantum mechanics. The main result of this paper is the identification of the internal degrees of freedom e and m as constant external fields. It suggests that the next item to study, after the one-particle model, is not the field version of the model, but the interaction of a single particle with varying and fully quantized external fields.

An important difference from DFR is that we consider not only noncommuting spacetime coordinates but also noncommuting momentum operators. This is a deliberate choice. It is made possible by considering representations which are also projective for shifts in spacetime, the generators of which are (proportional to) the momentum operators. The consequences of making this choice will become clear further on. While considering these projective representations it turns out to be obvious to allow the metric tensor g to depend on the internal degrees of freedom e and m. We use the notation $\gamma(e, m)$ for this e, m-dependent metric tensor while g always denotes the metric tensor $[1, -1, -1, -1]$ of Minkowski space.

Another modification to the model is the interchange of the two internal degrees of freedom e and m (corrected by a factor $e \cdot m$ to restore time reversal symmetry). This intervention is needed to allow for the interpretation of the internal degrees of freedom e and m as (analogues of) electric and magnetic fields. Finally, the latter interpretation suggests the introduction of a coupling constant λ and of charges a and b.

2. The model

The internal degrees of freedom consist of two vectors e and m in $\boldsymbol{R}^3$ satisfying $|e| = |m|$ and $e \cdot m = \pm 1$. These e, m-pairs are the points of the internal configuration space Σ. It consists of two subspaces Σ_+ and Σ_- corresponding with the two possible signs of the scalar product $e \cdot m$. The DFR model [5, 7] give an extensive justification of this model. For our purposes it is important that under Lorentz transformations points of Σ transform into themselves. These transformations are defined as follows. Given a point e, m in Σ introduce the following anti-symmetric matrix:

$$\epsilon(e, m) = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix}. \tag{5}$$

Let Λ be a Lorentz transformation. The transformation of $\epsilon(e, m)$ using Λ is denoted $\epsilon(e', m')$

$$\epsilon(e', m') = \Lambda^{-1}\epsilon(e, m)\tilde{\Lambda}^{-1}. \tag{6}$$

It is again an anti-symmetric matrix. It is not difficult to show that e', m' is again a point of Σ. Hence, the Lorentz transformation Λ maps the point e, m into the point e', m'. Note that (6) differs from the conventions used in [11]. These differences are necessary because of the swap of meaning of e and m.

In the DFR paper the variables e and m are by definition the entries of the four-by-four anti-symmetric matrix appearing in the commutation relations

$$[Q_\mu, Q_\nu] = \mathrm{i} l_{\mathrm{P}}^2 \mathcal{Q}_{\mu,\nu} \tag{7}$$

(l_{P} is Planck's length). In our notations this means that $\mathcal{Q} = \epsilon(e, m)$. Our actual result gives $\mathcal{Q}$ proportional to $\epsilon^{-1}(e, m)$. Note that the inverse of the matrix $\epsilon(e, m)$ is given by

$$\epsilon^{-1}(e, m) = -(e \cdot m)\epsilon(m, e) \tag{8}$$

so that again the differences are explained by the interchange of e and m.

In what follows we need to integrate over Σ in a covariant way. This integration is defined by

$$\int_\Sigma \mathrm{d}e\,\mathrm{d}m\, f(e, m) = \int_{\boldsymbol{R}^6} \mathrm{d}e\,\mathrm{d}m\,\delta((e \cdot m)^2 - 1)\delta(|e|^2 - |m|^2) f(e, m). \tag{9}$$

It is straightforward to show that the transformation of $\boldsymbol{R}^6$, defined by (6), has determinant ± 1. As a consequence the integration in the rhs of (9) is invariant under proper Lorentz transformations.

3. A fundamental length

Many authors have proposed that at very short distances the coordinates of spacetime should be discrete, or that at least Heisenberg-type uncertainty relations should hold for time and position operators. The argument is that at the scale of Planck's length

$$l_{\mathrm{P}} = \sqrt{G\hbar c^{-3}} \tag{10}$$

the quantum nature of gravitational forces is important and changes the structure of spacetime. Once one accepts the relevance of the fundamental unit of length l_{P} all distances can be expressed as dimensionless numbers. In particular, one can convert inverse lengths to lengths. Using Planck's constant $\hbar$ one can then convert momenta into lengths. In what follows we will use this idea of an absolute length l to convert shifts in position q into shifts in wavevector k by means of the relation $k = l^{-2}q$. However, this formula does not behave correctly under time reversal. In the present model we can correct for this by multiplying by the scalar product $e \cdot m$, which changes sign under time reversal, i.e. $(e \cdot m)l^{-2}q$ behaves as a wavevector (it transforms as a pseudo-vector).

As early as 1949 Born [1] suggested that, in analogy with the rest mass squared given by

$$c^{-2} \sum_{\mu,\nu=0}^{3} g_{\mu,\nu} p_\mu p_\nu \tag{11}$$

also the pseudo-distance

$$d(q, q') = \sum_{\mu,\nu=0}^{3} g_{\mu,\nu}(q - q')_\mu (q - q')_\nu \tag{12}$$

could have a discrete spectrum. He proposed to introduce a new pseudo-metric, which in our notations reads

$$\sum_{\mu,\nu} g_{\mu,\nu}(q_\mu q_\nu + l^4 k_\mu k_\nu). \tag{13}$$

The group of symmetries leaving this pseudo-metric invariant is larger than the Poincaré group. By requiring covariance for this larger group extra constraints are added to the theory. (See [8].) It is straightforward to see that our analysis of the DFR model can be extended to include this larger group. However, in this paper we restrict ourselves to the requirement of Poincaré invariance. Because the model contains a fixed length there is no way to extend the Poincaré group with dilatations to obtain the Weyl group.

4. Correlation function approach

The commutation relations (7) are the basis of the DFR paper. Here, the starting point is a correlation function denoted $F(f; k, q; k', q')$, with $f(e, m)$ any function of e and m, and with k, k', q and q' four-vectors (k has the meaning of a shift in the space of wavevectors, and q of a shift in spacetime). Later on we construct a Hilbert space representation which is such that

$$F(f; k, q; k', q') = \langle\psi|U(k', q')\hat{f}U(k, q)^\dagger|\psi\rangle \tag{14}$$

holds. In this expression ψ is a wavefunction, $U(k, q)$ is a projective unitary representation of the additive group $\boldsymbol{R}^4 \times \boldsymbol{R}^4$ of shifts in spacetime and in wavevector space, and $\hat{f}$ is the quantization of the function $f(e, m)$.

The technique of constructing quantum systems starting not from commutation relations but from correlation functions has been developed recently in a mathematical paper [11]. It is a generalization of the C^*-algebraic approach, which requires an algebraic structure together with correlation functions determining the state of the system. In the new approach, the C^*-algebra is replaced by a group of symmetries X acting on 'classical' functions, e.g. functions of the position of the particle, or, as we do here, functions $f(e, m)$ of the internal degrees of freedom e and m. One of the advantages of the formalism is the room it leaves for projective representations of X. This point is crucial for the present paper.

We need an explicit expression of $F(f; k, q; k', q')$ in closed form. Typically, this kind of correlation function, which can be expressed in closed form, describes coherent states and has a Gaussian form. Our *ansatz* is

$$F(f; k, q; k', q') = \int_\Sigma \mathrm{d}e\, \mathrm{d}m\, w(e, m) f(e, m) \xi(k, q; k', q'; e, m) \times \exp\left(-\frac{1}{2\lambda} s(k, q; k', q'; e, m)\right). \tag{15}$$

This expression has been obtained by elaborating the simpler versions found in [10, 11]. In this expression $w(e, m)$ is a density function, i.e. $w(e, m)$ is positive and normalized

$$\int_\Sigma \mathrm{d}e\, \mathrm{d}m\, w(e, m) = 1 \tag{16}$$

$\xi(k, q; k', q'; e, m)$ is a complex phase factor, λ is a coupling constant discussed later on and $s(k, q; k', q'; e, m)$ is a real function, bilinear in k, q and k', q'. In order to be a correlation function (15) should satisfy conditions of positivity, normalization, covariance and continuity (see [11]). The explicit choice of $\xi(k, q; k', q'; e, m)$ and $s(k, q; k', q'; e, m)$, made below, satisfies these requirements.

The phase factor $\xi(k, q; k', q'; e, m)$ is written in the following way:

$$\xi(k, q; k', q'; e, m) = \exp\left(\frac{\mathrm{i}}{2\lambda}(e \cdot m)\, u \epsilon^{-1}(e, m) u'\right) \tag{17}$$

where

$$u = lk + \lambda l^{-1} \eta(e, m) q \qquad \text{and} \qquad u' = lk' + \lambda l^{-1} \eta(e, m) q'.$$

It involves the four-by-four matrix η given by

$$\eta(e, m) = (e \cdot m)\, \epsilon(e, m) \gamma^{-1}(e, m). \tag{18}$$

Its function is to transform positions into wavevectors. As discussed before, the factor $(e \cdot m)$ is necessary because wavevectors are pseudovectors changing sign under time reversal. The choice of η has to be made in such a way that

$$\eta(e', m') = \Lambda^{-1} \eta(e, m) \Lambda \tag{19}$$

holds for any proper Lorentz transformation Λ, when e', m' are related to e, m via (6). This condition is satisfied if the matrix $\gamma(e, m)$ transforms like ϵ, i.e.

$$\gamma(e', m') = \Lambda^{-1} \gamma(e, m) \tilde{\Lambda}^{-1} \tag{20}$$

should hold. Note that $\gamma(e, m) = g$ satisfies the latter condition. Throughout this paper one can substitute $\gamma(e, m)$ by g. Note that we assume in the following that $\gamma(e, m)$ is a symmetric matrix.

The function $s(k, q; k', q'; e, m)$ is given by

$$s(k, q; k', q'; e, m) = (u - u') T(e, m) (u - u') \tag{21}$$

with u and u' as in (17). It involves a symmetric four-by-four matrix $T_{\mu,\nu}(e, m)$, whose elements may depend on e and m. At first sight the expression $\exp\left(-(1/2\lambda) s(k, q; k', q'; e, m)\right)$ does not appear Lorentz covariant. It is indeed necessary to make a special 'covariant' choice of the matrix $T_{\mu,\nu}(e,m)$. The requirement of covariance turns out to be that it should transform in the same way as $\epsilon(e,m)$. This means that, if the Lorentz transformation Λ transforms $\epsilon(e,m)$ into $\epsilon(e',m')$, then also

$$T(e',m') = \Lambda^{-1}T(e,m)\tilde{\Lambda}^{-1} \tag{22}$$

holds. Assume for example that $T(e,m) = (1/2)\boldsymbol{I}$ (half the identity matrix) whenever the length of e and m is equal to unity. Next define $T(e,m)$ for arbitrary e and m by $T(e,m) = (1/2)\Lambda^{-1}\tilde{\Lambda}^{-1}$, where Λ is any Lorentz boost for which $\epsilon(e,m) = \Lambda^{-1}\epsilon(e_0,m_0)\tilde{\Lambda}^{-1}$ with e_0 and m_0 vectors of unit length.

5. Hilbert space representation

The correlation function (15) can be used to define a scalar product for wavefunctions of the form $\psi(k,q,e,m)$ by the formula

$$\begin{aligned}\langle\psi|\phi\rangle = \int_\Sigma \mathrm{d}e\,\mathrm{d}m \int_{\boldsymbol{R}^4}\mathrm{d}k \int_{\boldsymbol{R}^4}\mathrm{d}q \int_{\boldsymbol{R}^4}\mathrm{d}k' \int_{\boldsymbol{R}^4}\mathrm{d}q' \\ \times\phi(k,q,e,m)\overline{\psi(k',q',e,m)}\xi(k,q;k',q';e,m) \\ \times\exp\left(-\frac{1}{2\lambda}s(k,q;k',q';e,m)\right).\end{aligned} \tag{23}$$

This scalar product defines the Hilbert space of wavefunctions. We cannot use the more common representation involving square integrable wavefunctions. Therefore one should be careful with the traditional interpretation of $|\psi(k,q,e,m)|^2$ as a probability density.

In this Hilbert space exists a unitary representation of shifts in k- and q-space. It is given by

$$U(k,q)\psi(k',q',e,m) = \psi(k+k',q+q',e,m)\xi(k',q';k,q;e,m). \tag{24}$$

The representation is projective. Indeed, one verifies immediately, using (24), that

$$U(k,q)U(k',q') = \hat{\xi}(k,q;k',q')U(k+k',q+q'). \tag{25}$$

We use a $\hat{}$ to denote multiplication operators. So, if f is a function of k,q,e,m then $\hat{f}$ is the operator which multiplies $\psi(k,q,e,m)$ with $f(k,q,e,m)$. In particular, $\hat{\xi}(k',q';k'',q'')$ is the operator which multiplies $\psi(k,q,e,m)$ with $\xi(k',q';k'',q'';e,m)$.

The correlation functions $F(f;k,q;k',q')$ follow from equations (14) and (23) if the wavefunction ψ is taken as

$$\psi(k,q,e,m) = \delta(k)\delta(q)\sqrt{w(e,m)} \tag{26}$$

with $\delta(k)$ and $\delta(q)$ Dirac's delta function. Remember that the wavefunctions are *not* necessarily square integrable functions, so the choice (26) is acceptable. On the other hand, the interpretation of $|\psi(k,q,e,m)|^2$ as a probability density of finding the quantum particle in the state k,q,e,m is *not* correct. This will be clear from the explicit expression for position and momentum operators as given in the next section.

6. Position and momentum operators

The position and momentum operators Q_μ and $P_\mu = \hbar K_\mu$ are by definition the generators of the group of shifts in wavevector space and spacetime respectively. Let us fix conventions in such a way that

$$U(k,q) = \exp(-\mathrm{i}k\hat{\gamma}^{-1}Q + \mathrm{i}\hat{\gamma}^{-1}qK) \tag{27}$$

holds. A quick calculation using (24) gives then a result which can be written as

$$\begin{aligned} Q_\mu &= \sum_{\nu=0}^{3} \hat{\gamma}_{\mu,\nu} \mathrm{i} \frac{\partial}{\partial k_\nu} + \frac{1}{2}\hat{q}_\mu - \frac{al^2}{\hbar c}\hat{\Omega}_\mu \\ K_\mu &= -\sum_{\nu=0}^{3} \hat{\gamma}_{\mu,\nu} i \frac{\partial}{\partial q_\nu} + \frac{1}{2}\hat{k}_\mu - \frac{b}{\hbar c}\sigma_3 \hat{A}_\mu \end{aligned} \tag{28}$$

with σ_3 the operator which multiplies the wavefunction $\psi(k, q, e, m)$ by $e \cdot m$, with a and b 'charges' of the particle, with Ω_μ given by

$$\Omega_\mu(k, e, m) = -\frac{\hbar c}{2\lambda a} \sum_{\nu=0}^{3} \eta^{-1}_{\mu,\nu}(e, m) k_\nu \tag{29}$$

and with A_μ given by

$$A_\mu(q, e, m) = -(e \cdot m)\, \frac{\lambda \hbar c}{2bl^2} \sum_{\nu=0}^{3} \eta_{\mu,\nu}(e, m) q_\nu. \tag{30}$$

The quantities $A_\mu(q, e, m)$ form a vector potential. They satisfy the rather unusual condition

$$\sum_{\mu,\nu} \gamma^{-1}_{\mu,\nu}(e, m) q_\mu A_\nu(q, e, m) = -\frac{\lambda}{b}\frac{\hbar c}{2l^2} \gamma^{-1}(e, m) q \epsilon(e, m) \gamma^{-1}(e, m) q = 0. \tag{31}$$

Introduce the notations

$$E_\alpha = \sum_\nu \gamma_{0,\nu}(e, m) \frac{\partial A_\alpha}{\partial q_\nu} - \sum_\nu \gamma_{\alpha,\nu}(e, m) \frac{\partial A_0}{\partial q_\nu} \qquad \alpha = 1, 2, 3 \tag{32}$$

and

$$B_\alpha = -\sum_{\nu=0}^{3} \sum_{\beta,\zeta=1}^{3} \overline{\varepsilon}_{\alpha,\beta,\zeta} \gamma_{\zeta,\nu}(e, m) \frac{\partial A_\beta}{\partial q_\nu} \qquad \alpha = 1, 2, 3 \tag{33}$$

with $\overline{\varepsilon}_{\alpha,\beta,\zeta}$ the fundamental anti-symmetric tensor of dimension three. One calculates

$$\begin{aligned} E_\alpha &= -(e \cdot m)\, \frac{\lambda}{b}\frac{\hbar c}{2l^2} \left(\sum_\nu \gamma_{0,\nu} \eta_{\alpha,\nu} - \sum_\nu \gamma_{\alpha,\nu} \eta_{0,\nu} \right) \\ &= \frac{\lambda}{b}\frac{\hbar c}{l^2} e_\alpha \end{aligned} \tag{34}$$

and

$$\begin{aligned} B_\alpha &= (e \cdot m) \frac{\lambda}{b}\frac{\hbar c}{2l^2} \sum_{\nu=0}^{3} \sum_{\beta,\zeta} \overline{\varepsilon}_{\alpha,\beta,\zeta} \gamma_{\zeta,\nu} \eta_{\beta,\nu} \\ &= \frac{\lambda}{b}\frac{\hbar c}{l^2} m_\alpha. \end{aligned} \tag{35}$$

Assume now that λ is the fine-structure constant of electromagnetism and that b is the charge of the proton. They are related by

$$b^2 = \lambda \hbar c. \tag{36}$$

Then the equations (34) and (35) become

$$\begin{aligned} E_\alpha &= \frac{b}{l^2} e_\alpha \\ B_\alpha &= \frac{b}{l^2} m_\alpha. \end{aligned} \tag{37}$$

Note that b/l^2 is the strength of the electric field of the proton at distance l. One concludes that e_α and m_α can be interpreted as a magnetic and an electric field respectively, measured in absolute units, which relate to the elementary charge b and the intrinsic length l.

In analogy with (31), the $\Omega_\mu(k, e, m)$ satisfy the condition

$$\sum_\mu \gamma^{-1}_{\mu,\nu}(e,m)k_\mu\Omega_\nu(k,e,m) = -\frac{\hbar c}{2\lambda a}\sum_{\mu,\nu,\zeta}\gamma^{-1}_{\mu,\nu}(e,m)k_\mu\eta^{-1}(e,m)_{\nu,\zeta}k_\zeta$$
$$= -\frac{\hbar c}{2\lambda a}(e\cdot m)k\epsilon^{-1}(e,m)k = 0. \tag{38}$$

The fields E_α and B_α can be obtained from $\Omega_\mu(k, e, m)$ by

$$\begin{aligned} E_\alpha &= \frac{\lambda a}{l^2 b}\sum_{\beta,\zeta=1}^{3}\sum_{\nu=0}^{3}\overline{\varepsilon}_{\alpha,\beta,\zeta}\gamma^{-1}_{\zeta,\nu}(e,m)\frac{\partial\Omega_\nu}{\partial k_\beta} \\ B_\alpha &= \frac{\lambda a}{l^2 b}\left[\sum_{\nu=0}^{3}\gamma^{-1}_{0,\nu}(e,m)\frac{\partial\Omega_\nu}{\partial k_\alpha} - \sum_{\nu=0}^{3}\gamma^{-1}_{\alpha,\nu}(e,m)\frac{\partial\Omega_\nu}{\partial k_0}\right] \end{aligned} \tag{39}$$

($\alpha = 1, 2, 3$). The symmetry between these relations and (32) and (33) can be understood because the matrices $\frac{\partial\Omega_\mu}{\partial k_\nu}$ and $\frac{\partial A_\mu}{\partial q_\nu}$ are each other's inverses (up to a constant factor). Indeed, one has

$$\sum_{\nu=0}^{3}\frac{\partial A_\mu}{\partial q_\nu}\frac{\partial\Omega_\nu}{\partial k_\sigma} = (e\cdot m)\frac{\hbar^2 c^2}{4abl^2}\delta_{\mu,\sigma} \tag{40}$$

with $\delta_{\mu,\sigma}$ Kronecker's delta.

7. Commutation relations

From (28) one obtains the following commutation relations:

$$\begin{aligned}\left[Q_\mu, Q_\nu\right] &= -\frac{al^2}{\hbar c}\sum_{\sigma=0}^{3}\left[\mathrm{i}\frac{\partial}{\partial k_\sigma}, \hat\gamma_{\mu,\sigma}\hat\Omega_\nu - \hat\gamma_{\nu,\sigma}\hat\Omega_\mu\right] \\ &= \mathrm{i}\frac{l^2}{2\lambda}\sum_{\sigma=0}^{3}\left(\hat\gamma_{\mu,\sigma}\hat\eta^{-1}_{\nu,\sigma} - \hat\gamma_{\nu,\sigma}\hat\eta^{-1}_{\mu,\sigma}\right) \\ &= -\mathrm{i}\frac{l^2}{\lambda}\sigma_3(\hat\gamma\hat\epsilon^{-1}\hat\gamma)_{\mu,\nu} \end{aligned} \tag{41}$$

$$\begin{aligned}\left[K_\mu, K_\nu\right] &= \frac{\lambda}{\hbar c}\sigma_3\sum_{\sigma=0}^{3}\left[\mathrm{i}\frac{\partial}{\partial q_\sigma}, \hat\gamma_{\mu,\sigma}\hat A_\nu - \hat\gamma_{\nu,\sigma}\hat A_\mu\right] \\ &= -\mathrm{i}\frac{\lambda}{2l^2}\sum_{\sigma=0}^{3}\left(\hat\gamma_{\mu,\sigma}\hat\eta_{\nu,\sigma} - \hat\gamma_{\nu,\sigma}\hat\eta_{\mu,\sigma}\right) \\ &= \mathrm{i}\frac{\lambda}{l^2}\sigma_3\hat\epsilon_{\mu,\nu} \end{aligned} \tag{42}$$

and

$$\left[K_\mu, Q_\nu\right] = -\mathrm{i}\hat\gamma_{\mu,\nu}. \tag{43}$$

As explained before, the main difference between (41) and (7) arises from the interchange of e and m. Further differences are the appearance in (41) of the inverse of the coupling constant

λ and of the generalized metric tensor $\gamma(e, m)$. If $\gamma(e, m) \equiv g$ then the only effect is a change of sign for the commutator between the time operator and the position operators. The appearance of factors $\gamma(e, m)$ in (41) and (43) is a consequence of including $\gamma^{-1}(e, m)$ in the definition (24) of the generators K_μ and Q_μ.

Many authors, e.g. [4, 6], have studied noncanonical commutation relations comparable with (41)–(43)—see for example the references cited in [5, 7]. A review of these works is outside the scope of this paper.

Note that, if one takes $\hat{\Omega}_\mu$ and $\hat{A}_\mu$ equal to zero in (28), then one obtains a representation describing a particle of mass zero in the off-shell formalism of relativistic quantum mechanics. The noncanonical commutation relations, which we have here, are a consequence of the presence in (28) of terms containing $\hat{\Omega}_\mu$ and $\hat{A}_\mu$ respectively. Now, the procedure of replacing momenta P_μ by new momenta $P_\mu - (b/c)A_\mu$ is well known from electrodynamics. Note that the components of the new momenta $P_\mu - (b/c)A_\mu$ do not necessarily commute between themselves (this fact is well known, and was used e.g. in [4] as an argument to introduce noncommuting position operators). Hence noncommuting momenta are quite common in quantum electrodynamics. In the present model there is not only a substitution of P_μ by $P_\mu - \sigma_3(b/c)\hat{A}_\mu$ but also a substitution of Q_μ by $Q_\mu - (al^2/\hbar c)\hat{\Omega}_\mu$. The latter is responsible for the nonvanishing time–position commutators.

Because we assume that a fixed length exists in the model, we can add positions and wavevectors, as in (17). A similar combination can be made on the level of operators. Introduce operators X_μ and Y_μ by

$$\begin{aligned} X_\mu &= lK_\mu + \lambda l^{-1} \sum_\nu \eta_{\mu,\nu} Q_\nu \\ Y_\mu &= lK_\mu - \lambda l^{-1} \sum_\nu \eta_{\mu,\nu} Q_\nu. \end{aligned} \tag{44}$$

The X_μ satisfy essentially the same commutation relations as the K_μ

$$\left[X_\mu, X_\nu\right] = 4\mathrm{i}\lambda\sigma_3\hat{\epsilon}_{\mu,\nu} \tag{45}$$

while the Y_μ commute between themselves and with the X_μ

$$\left[Y_\mu, Y_\nu\right] = 0 \qquad \text{and} \qquad \left[Y_\mu, X_\nu\right] = 0. \tag{46}$$

The substitution of momenta and position operators by operators with subtracted vector potentials, as discussed in the previous paragraph, translate into a substitution of X_μ by $X_\mu + (1/2)u_\mu$, with u_μ as in (17). Hence, the origin of noncanonical commutation relations in the present model is a constant external field which couples with the generators X_μ. Because the latter are a linear combination of position and momentum operators both pick up noncommuting terms. The Y_μ are not influenced by the external field and commute with the shifts in position or momentum space. However, they do not commute with Lorentz transformations (see the next section). It is therefore clear that the Y_μ describe internal degrees of freedom of the particle. On the sub-Hilbert space $\mathcal{H}_{\mathrm{ext}}$ of wavefunctions which depend only on u, e and m, the position and momentum operators can be written as

$$\begin{aligned} l^{-1} Q_\mu\big|_{\mathcal{H}_{\mathrm{ext}}} &= \sum_{\nu=0}^{3} \hat{\gamma}_{\mu,\nu}\mathrm{i}\frac{\partial}{\partial u_\nu} + \frac{1}{2\lambda}\sum_{\nu=0}^{3} \hat{\eta}^{-1}_{\mu,\nu}\hat{u}_\nu \\ lK_\mu\big|_{\mathcal{H}_{\mathrm{ext}}} &= \lambda\sigma_3 \sum_{\nu=0}^{3} \hat{\epsilon}_{\mu,\nu} i\frac{\partial}{\partial u_\nu} + \frac{1}{2}\hat{u}_\mu \end{aligned} \tag{47}$$

so that

$$X_\mu\big|_{\mathcal{H}_{\rm ext}} = 2\lambda\sigma_3\sum_{\nu=0}^{3}\hat\epsilon_{\mu,\nu}\mathrm{i}\frac{\partial}{\partial u_\nu}+\hat u_\mu \qquad (48)$$
$$Y_\mu\big|_{\mathcal{H}_{\rm ext}} = 0.$$

Hence in this subspace position and momentum operators coincide.

8. Poincaré invariance

Shifts of the particle in spacetime are described by the unitary operators $U(0,q)$. Indeed, one verifies that

$$U(0,q)Q_\mu U(0,q)^\dagger = Q_\mu + q_\mu. \qquad (49)$$

On the other hand

$$U(0,q)K_\mu U(0,q)^\dagger = K_\mu + \frac{\lambda}{l^2}\sum_\nu \hat\eta_{\mu,\nu}q_\nu. \qquad (50)$$

Clearly, the operators K_μ are not conserved under shifts in spacetime. This is understandable because the particle moves in external fields.

Similarly, shifts in the space of wavevectors are described by the unitary operators $U(k,0)$. Indeed, one has

$$U(k,0)K_\mu U(k,0)^\dagger = K_\mu + k_\mu. \qquad (51)$$

Next we define a unitary representation R of the proper Lorentz group. The *ansatz* is

$$R(\Lambda)\psi(k,q,e,m) = \psi(\Lambda^{-1}k, \Lambda^{-1}q, e', m') \qquad (52)$$

with e', m' related to e, m by (6). The conjugate operator $R(\Lambda)^\dagger$ is given by

$$R(\Lambda)^\dagger\psi(k,q,e',m') = \psi(\Lambda k, \Lambda q, e, m). \qquad (53)$$

It is now straightforward to verify that $R(\Lambda)$ is a unitary representation of the proper Lorentz group.

We cannot use (52) for the whole of the Lorentz group because time reversal must be implemented as an anti-unitary operator [2]. Indeed, under time reversal q_μ goes into $-g_{\mu,\mu}q_\mu$ while k_μ goes into $g_{\mu,\mu}k_\mu$. The operator Θ given by

$$\Theta\psi(k,q,e,m) = \overline{\psi(gk,-gq,-e,m)} \qquad (54)$$

satisfies all requirements. It obviously satisfies $\Theta^2 = \boldsymbol{I}$ and one verifies that

$$(\Theta\phi,\psi) = (\Theta\psi,\phi). \qquad (55)$$

Finally, the parity operator P is defined as an isometry between Hilbert spaces by

$$P\psi(k,q,e,m) = \psi(gk,gq,-e,m). \qquad (56)$$

The parity-inverted scalar product is given by

$$\langle\psi|\phi\rangle' = \int_\Sigma \mathrm{d}e\,\mathrm{d}m \int_{\boldsymbol{R}^4}\mathrm{d}k\int_{\boldsymbol{R}^4}\mathrm{d}q\int_{\boldsymbol{R}^4}\mathrm{d}k'\int_{\boldsymbol{R}^4}\mathrm{d}q'$$
$$\times\phi(k,q,e,m)\overline{\psi(k',q',e,m)}\xi(gk,gq;gk',gq';-e,m)$$
$$\times\exp\left(-\frac{1}{2\lambda}s(gk,gq;gk',gq';-e,m)\right). \qquad (57)$$

It satisfies

$$\langle P\psi|P\phi\rangle' = \langle\psi|\phi\rangle. \qquad (58)$$

9. Invariants

The position and wavevector operators Q_μ and K_μ transform as expected under proper Lorentz transformations. From (52), (53) and the definitions (28) one obtains

$$\begin{aligned} R(\Lambda)Q_\mu R(\Lambda)^\dagger &= \sum_\nu \Lambda^{-1}_{\mu,\nu} Q_\nu \\ R(\Lambda)K_\mu R(\Lambda)^\dagger &= \sum_\nu \Lambda^{-1}_{\mu,\nu} K_\nu. \end{aligned} \tag{59}$$

Note that also

$$R(\Lambda)\hat{\gamma}_{\mu,\nu}R(\Lambda)^\dagger = (\Lambda^{-1}\hat{\gamma}\tilde{\Lambda}^{-1})_{\mu,\nu}. \tag{60}$$

Introduce the squared mass operator M^2 by

$$c^2\hbar^{-2}M^2 = \sum_{\mu,\nu} \hat{\gamma}^{-1}_{\mu,\nu} K_\mu K_\nu. \tag{61}$$

Then one has obviously

$$R(\Lambda)M^2R(\Lambda)^\dagger = M^2. \tag{62}$$

Note that M^2 is not necessarily invariant under shifts in spacetime.

Similarly, the squared eigentime operator

$$\sum_{\mu,\nu} \hat{\gamma}^{-1}_{\mu,\nu} Q_\mu Q_\nu \tag{63}$$

is also invariant under proper Lorentz transformations.

10. Gauge transformations

Consider the gauge transformation

$$\begin{aligned} A_\mu &\to A'_\mu = A_\mu + \sum_\nu \gamma_{\mu,\nu}\frac{\partial\chi}{\partial q_\nu} \\ \Omega_\mu &\to \Omega'_\mu = \Omega_\mu - (e\cdot m)\frac{2b^2}{\hbar l^2}\sum_\nu \gamma_{\mu,\nu}\frac{\partial\chi}{\partial k_\nu} \end{aligned} \tag{64}$$

with χ an arbitrary function of k, q, e, m. Under this transformation E_α and B_α, as given by (32) and (33), are invariant. Also the commutation relations (41)–(43) are invariant. Now let

$$U'(k,q) = \exp(-\mathrm{i}k\hat{\gamma}^{-1}Q' + \mathrm{i}\hat{\gamma}^{-1}qK') \tag{65}$$

with Q'_μ and K'_μ derived from (28) by substituting A_μ by A'_μ. Then U' is again a projective representation of the covariance system. It involves the same operator-valued phase factor $\xi(k,q;k',q')$ because the latter depends only on the commutation relations (41)–(43).

Fix a positive number κ, which is the rest mass of the particle multiplied by $c/\hbar$. Assume that $\psi(k,q,e,m)$ is a solution of the eigenvalue problem

$$(c/\hbar)^2M^2\psi = \kappa^2\psi \tag{66}$$

(we do not assume that ψ is a wavefunction belonging to the Hilbert space with scalar product (23)). Then the function $\psi'(k,q,e,m)$ given by

$$\psi'(k,q,e,m) = \exp\left(\mathrm{i}\frac{b}{\hbar c}(e\cdot m)\chi(k,q,e,m)\right)\psi(k,q,e,m) \tag{67}$$

is a solution of the eigenvalue problem

$$(c/\hbar)^2(M')^2\psi' = \kappa^2\psi'. \tag{68}$$

This property is what one understands by gauge invariance of the model. In order to check that (68) holds let us calculate

$$\begin{aligned}(\hbar/c)^2\kappa^2\psi' &= \exp\left(\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)(\hbar/c)^2\kappa^2\psi\\ &= \exp\left(\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M^2\psi\\ &= \exp\left(\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M^2\exp\left(-\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)\psi'.\end{aligned} \tag{69}$$

Now use that

$$K_\nu\exp\left(-\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right) = \exp\left(-\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)K'_\nu \tag{70}$$

to obtain

$$M^2\exp\left(-\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right) = \exp\left(-\mathrm{i}\frac{b}{\hbar c}\sigma_3\hat{\chi}\right)M'^2 \tag{71}$$

so that (68) follows.

11. Discussion

We have shown in this paper that the variables e and m appearing in the DFR model can be explained as constant external fields. We have swapped the role of e and m so that e is an electric and m is a magnetic field vector. As a side effect we have also shown that the nonvanishing time–position commutators of the model arise by substituting the spacetime position operators Q_μ by $Q_\mu - (al^2/\hbar c)\Omega_\mu$, together with the well known substitution of momentum operators P_μ by $P_\mu - (b/c)A_\mu$. The vector potentials Ω_μ and A_μ are strictly linked because the field tensors $\frac{\partial\Omega_\mu}{\partial k_\nu}$ and $\frac{\partial A_\mu}{\partial q_\nu}$ are each other's inverses, up to a constant factor. Obviously, these findings are of interest in a more general context than that of this particular model. In the present model the k- and q-dependence of Ω_μ and A_μ respectively is trivial. In a more general context, we expect more complex dependence on k and q. In particular, nontrivial spacetime dependence of A_μ will lead to spacetime dependence of Ω_μ.

Projective representations with operator-valued phase factors play an important role in this paper. A more systematic study of this kind of representation is required. Also other aspects of the model require further investigation. In particular, we can make the following remarks.

- Throughout the paper the metric tensor g has been replaced by an operator $\hat{\gamma}$ because the mathematics allows us to do so. It is not clear what such an operator-valued metric tensor means.
- We did not consider spin of the particle. The particle/anti-particle structure of the model will be discussed in a subsequent paper.
- The covariant representation studied in this paper is reducible. Reducibility of the representation restricted to the sub-Hilbert space of wavefunctions depending on u, e and m (see the end of section 7) has not been investigated.

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